system but is dependent upon many different factors which may sometimes be opposing.[9] Thus Pauli and his pupils find minimum viscosity at an isoelectric point in protein sols due to minimum ionization of the protein. In the writer's gelatine experiments, however, the gelation or gelation viscosity of the gel was distinctly at a maximum at the isoelectric point due to maximum aggregation. It appears necessary to distinguish between these two kinds of viscosity.

Summary.—A close analogy to Osterhout's experiments on the electrical resistance of *Laminaria* is found in gelatine (plus NaOH), if we assume that the effect of time in the *Laminaria* experiments is to increase the concentrations of the salts in the cells of the tissue.

[1] These PROCEEDINGS, **2**, 534 (1916).

[2] For the sake of brevity the word precipitate is used throughout to denote not only an actual precipitate, but any accompanying conditions which vary with the amount of precipitate or the degree of precipitability.

[3] Cf. Osterhout, *Science*, **41**, 255 (1915) for summary of results.

[4] Pauli has concluded for other reasons that protoplasm reacts much like protein soils containing alkali. *Biochem. Zs.*, **24**, 239 (1910).

[5] Samec, *Koll.-Chem. Beihefte*, **5**, 141 (1913).

[6] Pauli, loc. cit.

[7] Osterhout, *Science*, **39**, 544 (1914).

[8] Spaeth, *Science*, **43**, 502 (1916).

[9] Ostwald, *Kolloid. Zs.*, **12**, 213 (1913).

ON CERTAIN ASYMPTOTIC EXPRESSIONS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS

By W. E. Milne

DEPARTMENT OF MATHEMATICS, BOWDOIN COLLEGE

Received by the Academy, July 6, 1916

The nature of the solutions of a certain linear differential equation containing a complex parameter has been investigated by Prof. G. D. Birkhoff,[1] who discovered the asymptotic character of the solutions when the parameter is large in absolute value. These results he employed in the study of expansion problems connected with the particular differential equation

$$\frac{d^n u}{dx^n} + P_2(x)\frac{d^{n-2}u}{dx^{n-2}} + \quad . \quad . \quad . \quad + P_n(x)\, u + \rho^n u = 0, \tag{1}$$

together with n linearly independent linear boundary conditions

$$W_1(u) = 0,\; W_2(u) = 0, \quad . \quad . \quad . \quad, W_n(u) = 0. \tag{2}$$

It is the aim of this paper to present asymptotic formulas for n linearly independent solutions $y_1, y_2, \ldots, y_n$ of equation (1) which are in

some respects more precise than those obtained by Birkhoff, and also to present similar asymptotic formulas for the n functions $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$, related to the n y's by the n identities

$$\sum_{i=1}^{n} y_i^{(k)} \bar{y}_i = \begin{cases} 0, & k = 0, \ldots, n-2, \\ 1, & k = n-1. \end{cases} \tag{3}$$

These functions $\bar{y}$ play an important rôle in the theory of linear differential equations. As is well known they form a system of linearly independent solutions of the equation adjoint to equation (1), while the expression

$$\sum_{i=1}^{n} y_i(x)\, \bar{y}_i(t)$$

is of fundamental importance in Lagrange's method of solving the non-homogeneous equation of which (1) is the reduced equation, as well as in the formation of the Green's function of the system (1) and (2). Asymptotic forms for the $\bar{y}$'s were also used by Birkhoff in his paper on expansion problems.[2]

I was led to make refinements in the forms of the y's and the $\bar{y}$'s in connection with a paper treating the degree of convergence of the expansion associated with the differential system (1) and (2).

Birkhoff divided the plane of the complex parameter ρ into $2n$ equal sectors,

$$S_k : k\pi/n \leqq \arg \rho \leqq (k+1)\pi/n, \quad k = 0, 1, \ldots, 2n-1,$$

and then numbered the n n-th roots of -1, $w_1, w_2, \ldots, w_n$, in such a manner that when ρ is on any given sector S_k, the inequalities

$$R(\rho w_1) \leqq R(\rho w_2) \leqq \ldots \leqq R(\rho w_n)$$

are satisfied, where $R(\rho w_i)$ denotes the real part of ρw_i. He then proved that if the coefficients $P_s(x)$ in (1) have continuous derivatives of all orders in the closed interval $a \leqq x \leqq b$, there exist for ρ in any given S_k n independent solutions of (1),

$$\begin{aligned} y_i &= u_i(x, \rho) + e^{\rho w_i (x-c)} E_{i0}/\rho^{m+1}, \\ y_i^{(k)} &= u_i^{(k)}(x, \rho) + e^{\rho w_i (x-c)} E_{ik}/\rho^{m+1-k}, \\ i = 1, 2, &\ldots, n; \; k = 1, 2, \ldots, n-1, \end{aligned} \tag{4}$$

in which

$$u_i(x, \rho) = e^{\rho w_i (x-c)} \left[1 + \frac{u_{i1}(x)}{\rho} + \ldots + \frac{u_{i\,m+1}(x)}{\rho^{m+1}} \right],$$

where m is any positive integer or zero.[3] The E_{ij} are functions of x and ρ which are bounded for x in (a, b) and for ρ in S_k and large in absolute value. The y_i are analytic in ρ in S_k, and the $u_{ij}(x)$ have derivatives of all orders with respect to x in (a, b).

The modification here proposed is this: *If the coefficients $P_s(x)$ have continuous derivatives of order $(m + n - s)$ in (a, b), m being any positive integer or zero, then there exist n solutions of (1) of the form (4), analytic in ρ in the sector S_k, and the functions $u_i(x,\rho)$ are of the form*

$$u_i(x, \rho) = e^{\rho w_i (x-c)} [1 + \varphi_1(x) / \rho w_i + \quad . \quad . \quad . \quad + \varphi_m(x) / (\rho w_i)^m],$$

where the functions $\varphi_j(x)$ have continuous derivatives of order $(m + n - j)$, and are independent of i.

The improvement in precision over Birkhoff's formulas consists primarily in putting the $u_i(x,\rho)$ in the form indicated, where the $\varphi_j(x)$ are independent of i; the details concerning the number of derivatives of the P's and the φ's are of secondary importance. A similar remark applies to the statement concerning the $\bar{y}$'s, which is as follows:

The n functions $\bar{y}_i$, determined by the n equations (3), have when $|\rho|$ is large the asymptotic form

$$\bar{y}_i = \frac{e^{-\rho w_i (x-c)}}{n (\rho w_i)^{n-1}} [v_i(x, \rho) + \bar{E}_i/\rho^{m+1}], \quad i = 1, 2, \quad . \quad . \quad . \quad , n,$$

where

$$v_i(x, \rho) = 1 + \psi_1(x) / \rho w_i + \quad . \quad . \quad . \quad + \quad \psi_m(x) / (\rho w_i)^m,$$

in which the functions $\psi_j(x)$ are independent of i and have continuous $(m + n - j)$-th derivatives in (a, b).

The proof of the asymptotic formulas for the y's is simply an adaptation of the proof given by Birkhoff. In the case of the $\bar{y}$'s, we verify the formulas by substituting the values of the $\bar{y}_i$ given above into equations (3) and showing that the ψ's and E's with the desired properties can be chosen to satisfy them.

[1] *Trans. Amer. Math. Soc.*, 9, 219–231, 373–395 (1908).

[2] Loc. cit., p. 391, formula (56).

[3] For the formulas here quoted see loc. cit., pp. 381–2, formulas (21) and (23). They are quoted in (4) with some slight changes.